Solid Abelian Groups

<u>Def</u> Given profinite $S = \varprojlim S_i$, define $\mathbb{Z}[S]^{\blacksquare} = \varprojlim \mathbb{Z}[S_i]$

Note: natural map $\mathbb{Z}[S] \to \mathbb{Z}[S]^{\blacksquare}$

A cond. ab. grp. A is <u>solid</u> if for all profinite S and $f: S \to A$

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[d] & A \\ \mathbb{Z}[S]^{\blacksquare} \arrow[ur, dashed, "\exists! \tilde{f}"'] & \end{tikzcd}$$

A complex $C \in D(\mathrm{Cond}(\mathrm{Ab}))$ is <u>solid</u> if

$$\mathrm{RHom}(\mathbb{Z}[S]^{\blacksquare}, C) \xrightarrow{\sim} \mathrm{RHom}(\mathbb{Z}[S], C)$$

<u>Rem</u>
- $A[0]$ solid $\Rightarrow$ A solid
- Spoiler: also $\Leftarrow$, more generally (requires proof)
  C solid $\Longleftrightarrow$ all $H^i(C)$ solid
- Spoiler: also "$\xrightarrow{\sim}$" for RHom
- Suffices to take S extremally disconnected.
  (by taking hypercovers)

<u>Goals</u> (I) Show $\mathbb{Z}[S]^{\blacksquare}$ is solid (free solid abelian group)

(II) Show $\mathrm{Solid} \overset{i}{\subseteq} \mathrm{Cond}(\mathrm{Ab})$ is abelian subcategory, and i has left adjoint $(-)^{\blacksquare}$ (solidification)

<u>Note</u> $\mathbb{Z}[S]^{\blacksquare} = \varprojlim \mathbb{Z}[S_i] = \varprojlim \underline{\mathrm{Hom}}(C(S_i,\mathbb{Z}),\mathbb{Z})$

$= \underline{\mathrm{Hom}}(\varinjlim C(S_i,\mathbb{Z}),\mathbb{Z}) = \underline{\mathrm{Hom}}(C(S,\mathbb{Z}),\mathbb{Z})$

In particular, $\mathbb{Z}[S]^{\blacksquare}(*) = \mathrm{Hom}(C(S,\mathbb{Z}),\mathbb{Z})$

$= \{\mathbb{Z}\text{-valued measures on } S\} = \mathcal{M}(S,\mathbb{Z})$

<u>Example</u> $S = \mathbb{N} \cup \{\infty\} = \varprojlim_n \{1,2,\dots,n\}$.

$C(S,\mathbb{Z})$ = eventually constant sequences

$\mathcal{M}(S,\mathbb{Z}) = \prod_{i\in\mathbb{N}} \mathbb{Z} \times \mathbb{Z}$

where $\underset{= \int f\mu}{\mu(f)} = \sum_{i\in\mathbb{N}} \mu_i \cdot (f(i) - f(\infty)) + \mu_\infty \cdot f(\infty)$.

$$\begin{tikzcd}
S \arrow[r, "f"] \arrow[d, "ev"'] & A & \int f\mu \\
\mathcal{M}(S,\mathbb{Z}) \arrow[ur, dashed, red] & & \\
\mu \arrow[uurr, mapsto, bend right] & &
\end{tikzcd}$$

- Non-archimedean !
- $\mathbb{R}$ not be solid.
- $\mathbb{Z}_p$ are solid.

Fact For S profinite, $C(S, \mathbb{Z}) \cong \bigoplus_I \mathbb{Z}$. (proven in notes does not use any condensed math)

Cor $\mathbb{Z}[S]^{\blacksquare} \cong \underline{\mathrm{Hom}}(C(S,\mathbb{Z}), \mathbb{Z}) \cong \mathrm{Hom}(\bigoplus_I \mathbb{Z}, \mathbb{Z})$

$$\cong \prod_I \underline{\mathrm{Hom}}(\mathbb{Z}, \mathbb{Z}) \cong \prod_I \mathbb{Z}.$$

Let's prove $\mathbb{Z}[S]^{\blacksquare}$ is solid.

Prop For any set I, $\prod_I \mathbb{Z}$ is solid as a complex, so also as abelian group.
(Same for $\mathbb{Z}[S]^{\blacksquare}$.)

Proof To show: $\prod_I R\mathrm{Hom}(\mathbb{Z}[T]^{\blacksquare}, \cancel{\prod_I}\mathbb{Z}) \cong \prod_I R\mathrm{Hom}(\mathbb{Z}[T], \cancel{\prod_I}\mathbb{Z})$

- RHS: $\mathrm{Ext}^i(\mathbb{Z}[T], \mathbb{Z}) = H^i(T, \mathbb{Z}) = \begin{cases} 0 & \text{if } i > 0 \text{ (} T \text{ profinite)} \\ \mathrm{Hom}(\mathbb{Z}[T], \mathbb{Z}) & \text{if } i = 0 \end{cases}$

$$\mathrm{Hom}(\mathbb{Z}[T], \mathbb{Z}) = C(T, \mathbb{Z}) \cong \bigoplus_J \mathbb{Z}.$$

$\Rightarrow$ RHS $= \bigoplus_J \mathbb{Z}$.

- LHS: Use $0 \to \prod_J \mathbb{Z} \to \prod_J \mathbb{R} \to \prod_J \mathbb{R}/\mathbb{Z} \to 0$, where $\prod_J \mathbb{Z} \cong \mathbb{Z}[T]^{\blacksquare}$

  - $R\mathrm{Hom}(\prod_J \mathbb{R}, \mathbb{Z}) \cong R\mathrm{Hom}_{\mathbb{R}}(\prod_J \mathbb{R}, \underbrace{R\underline{\mathrm{Hom}}(\mathbb{R}, \mathbb{Z})}_{0 \text{ by Lisanne}}) = 0$

  - $R\mathrm{Hom}(\prod_J \mathbb{R}/\mathbb{Z}) \underset{\text{Remy}}{\cong} \bigoplus_J \mathbb{Z}[-1]$

$\Rightarrow$ LHS $= \bigoplus_J \mathbb{Z}$. (If follow the maps, then see that map is identity.)

$\underbrace{\hspace{4cm}}_{\text{Used by Dion.}}$ $\square$

Thm (i) • $\mathrm{Solid} \subseteq \mathrm{Cond(Ab)}$ is abelian subcategory stable under all limits, colimits, extensions

• $\{\prod_I \mathbb{Z}\}$ form family of comp. proj. generators in Solid.

• Left adjoint $(-)^{■} \dashv i$

if $A = \mathrm{colim}\, \mathbb{Z}[S_i]$
then $A^{■} = \mathrm{colim}\, \mathbb{Z}[S_i]^{■}$

(ii) • $D(\mathrm{Solid}) \hookrightarrow D(\mathrm{Cond(Ab)})$ is fully faithful with essential image all solid complexes.

• C solid $\Leftrightarrow$ all $H^i(C)$ solid

• $(-)^{L■} \dashv i$ (Left derived solidification)

Strategy

① $\bigoplus \prod \mathbb{Z}$ are solid (as complex)

② Complexes $\cdots \to \bigoplus \prod \mathbb{Z} \to \cdots$ are solid

③ $\ker(\bigoplus \prod \mathbb{Z} \to \bigoplus \prod \mathbb{Z})$ are solid

④ $\mathrm{Solid} \subseteq \mathrm{Cond(Ab)}$ is abelian subcategory

① Can prove

② Truncation arguments

③ and ④ follow from derived category arguments.

①

Goal Show $M = \bigoplus \prod \mathbb{Z}$ is solid

i.e. $R\mathrm{Hom}(\mathbb{Z}[S]^{\blacksquare}, \underline{\bigoplus} \prod \mathbb{Z}) = R\mathrm{Hom}(\mathbb{Z}[S], \underline{\bigoplus} \prod \mathbb{Z})$

(want to pull out $\bigoplus$)

- RHS: take S extremally disconnected, then
  - $\mathbb{Z}[S]$ projective $\Rightarrow R\mathrm{Hom}(\mathbb{Z}[S], -) = \mathrm{Hom}(\mathbb{Z}[S], -)$
  - $\mathbb{Z}[S]$ compact $\Rightarrow \mathrm{Hom}(\mathbb{Z}[S], -)$ commutes with filtered colimits.

$\Rightarrow \mathrm{RHS} = \bigoplus R\mathrm{Hom}(\mathbb{Z}[S], \bigoplus \prod \mathbb{Z})$.

- LHS: Use $0 \to \prod_I \mathbb{Z} \to \prod_I \mathbb{R} \to \prod_I \mathbb{R}/\mathbb{Z} \to 0$.

Lemma $R\underline{\mathrm{Hom}}(\underbrace{\prod_I \mathbb{R}/\mathbb{Z}}_{T}, \bigoplus \prod \mathbb{Z}) = \bigoplus R\underline{\mathrm{Hom}}(\prod_I \mathbb{R}/\mathbb{Z}, \prod \mathbb{Z})$.

Proof Suffices to find resolution (extremally disconnected.)

$$\cdots \to \mathbb{Z}[S_1] \to \mathbb{Z}[S_0] \to T \to 0.$$

Take Breen-Deligne:

$$\begin{tikzcd}
T & \mathbb{Z}[T] \arrow[l, two heads] & \mathbb{Z}[T^2] \arrow[l] & \cdots \arrow[l] \\
& \mathbb{Z}[S_{0,0}] \arrow[u, two heads] & \mathbb{Z}[S_{1,0}] \arrow[l] \arrow[u, two heads] & \\
& \mathbb{Z}[S_{0,1}] \arrow[u] & \mathbb{Z}[S_{1,1}] \arrow[l] \arrow[u] &
\end{tikzcd}$$

Take total complex. $\square$

Lemma $R\underline{\mathrm{Hom}}(\mathbb{R}, \bigoplus \prod \mathbb{Z}) = 0 \quad \left(= \bigoplus R\underline{\mathrm{Hom}}(\mathbb{R}, \prod \mathbb{Z})\right)$

Proof Use $0 \to \mathbb{Z} \to \mathbb{R} \to \mathbb{R}/\mathbb{Z} \to 0$.
- $R\underline{\mathrm{Hom}}(\mathbb{Z}, M) = M$
- $R\underline{\mathrm{Hom}}(\mathbb{R}/\mathbb{Z}, \bigoplus \prod \mathbb{Z}) \overset{\text{Just proven}}{=} \bigoplus R\underline{\mathrm{Hom}}(\mathbb{R}/\mathbb{Z}, \prod \mathbb{Z})$

$\overset{\text{Remy}}{=} \bigoplus \prod \mathbb{Z}[-1] = M[-1]$ $\square$

Lemma $R\underline{\mathrm{Hom}}(\prod_I \mathbb{R}, M) = 0$

$\|$

Proof $R\underline{\mathrm{Hom}}_{\mathbb{R}}(\prod_I \mathbb{R}, \overbrace{R\underline{\mathrm{Hom}}(\mathbb{R}, M)}^{\text{previous lemma}}) = 0$. $\square$

③

<u>Proof</u> Take $f: Y \to Z$ with kernel K.

Pick resolution $\overbrace{\to \bigoplus \mathbb{Z}[S_j] \to \bigoplus \mathbb{Z}[S_i]}^{B} \to K \to 0$

Let $C = \left(\cdots \to \bigoplus \mathbb{Z}[S_j]^{\blacksquare} \to \bigoplus \mathbb{Z}[S_i]^{\blacksquare} \right)$

Now, $R\mathrm{Hom}(B, Y) = R\mathrm{Hom}(C, Y)$ and $R\mathrm{Hom}(B, Z) = R\mathrm{Hom}(C, Z)$

Hence,
$$\begin{tikzcd} B \arrow[r, "\sim"] \arrow[d] & K \\ C \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$
so B is a retract of C.

Finally,
$$R\mathrm{Hom}(\mathbb{Z}[T]^{\blacksquare}, \overset{B}{\cancel{K}}) = R\mathrm{Hom}(\mathbb{Z}[T], \overset{B}{\cancel{K}})$$

$\downarrow$

follows from $\cong$ of C
and B is a retract of C.

$\square$