

Solid Abelian Groups

Def Given profinite $S = \varprojlim S_i$, define $\mathbb{Z}[S]^{\square} = \varprojlim \mathbb{Z}[S_i]$
 Note: natural map $\mathbb{Z}[S] \rightarrow \mathbb{Z}[S]^{\square}$

A cond. ab. grp. A is solid if for all profinite S and $f: S \rightarrow A$

$$\begin{array}{ccc} S & \xrightarrow{f} & A \\ \downarrow & \nearrow & \uparrow \\ \mathbb{Z}[S]^{\square} & \xrightarrow{\exists! \tilde{f}} & A \end{array}$$

A complex $C \in D(\text{Cond}(\text{Ab}))$ is solid if

$$\text{RHom}(\mathbb{Z}[S]^{\square}, C) \xrightarrow{\sim} \text{RHom}(\mathbb{Z}[S], C)$$

- Rem
- $A[0]$ solid $\Rightarrow A$ solid
 - Später: also \Leftarrow , more generally C solid \Leftrightarrow all $H^i(C)$ solid (requires proof)
 - Später: also " $\xrightarrow{\sim}$ " for RHom
 - Suffices to take S extremally disconnected (by taking hypercovers)

- Goals
- (I) Show $\mathbb{Z}[S]^{\square}$ is solid (free solid abelian group)
- (II) Show $\text{Solid} \stackrel{i}{\subset} \text{Cond}(\text{Ab})$ is abelian subcategory, and i has left adjoint $(-)^{\square}$ (solidification)

Note $\mathbb{Z}[S]^{\text{fin}} = \varprojlim \mathbb{Z}[S_i] = \varprojlim \underline{\text{Hom}}(C(S_i, \mathbb{Z}), \mathbb{Z})$

$$= \underline{\text{Hom}}(\varinjlim C(S_i, \mathbb{Z}), \mathbb{Z}) = \underline{\text{Hom}}(C(S, \mathbb{Z}), \mathbb{Z})$$

In particular, $\mathbb{Z}[S]^{\text{fin}}(\ast) = \text{Hom}(C(S, \mathbb{Z}), \mathbb{Z})$

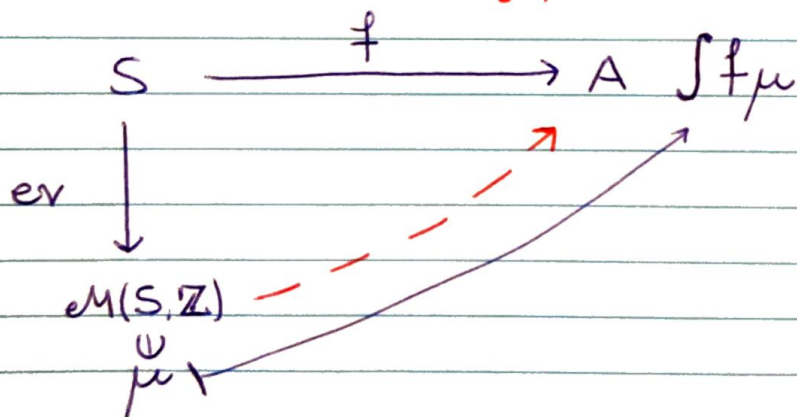
$$= \{ \mathbb{Z}\text{-valued measures on } S \} = eM(S, \mathbb{Z})$$

Example $S = \mathbb{N} \cup \{\infty\} = \varprojlim_n \{1, 2, \dots, n\}$

$C(S, \mathbb{Z}) =$ eventually constant sequences

$$eM(S, \mathbb{Z}) = \prod_{i \in \mathbb{N}} \mathbb{Z} \times \mathbb{Z}$$

where $\mu(f) = \sum_{i \in \mathbb{N}} \mu_i \cdot (f(i) - f(\infty)) + \mu_\infty \cdot f(\infty)$
 $= \int f d\mu$



- Non-archimedean!
- \mathbb{R} not be solid.
- \mathbb{Z}_p are solid.

Fact For S profinite, $C(S, \mathbb{Z}) \cong \bigoplus_I \mathbb{Z}$. (proven in notes does not use any condensed math)

Cor $\mathbb{Z}[S]^{\text{ab}} \cong \underline{\text{Hom}}(C(S, \mathbb{Z}), \mathbb{Z}) \cong \text{Hom}(\bigoplus_I \mathbb{Z}, \mathbb{Z})$
 $\cong \prod_I \underline{\text{Hom}}(\mathbb{Z}, \mathbb{Z}) \cong \prod_I \mathbb{Z}$.

Let's prove $\mathbb{Z}[S]^{\text{ab}}$ is solid.

Prop For any set I , $\prod_I \mathbb{Z}$ is solid as a complex, so also as abelian group.
 (Same for $\mathbb{Z}[S]^{\text{ab}}$.)

Proof To show: $\prod_I \text{RHom}(\mathbb{Z}[T]^{\text{ab}}, \prod_I \mathbb{Z}) \cong \prod_I \text{RHom}(\mathbb{Z}[T], \prod_I \mathbb{Z})$

• RHS: $\text{Ext}^i(\mathbb{Z}[T], \mathbb{Z}) = H^i(T, \mathbb{Z}) = \begin{cases} 0 & \text{if } i > 0 \text{ (T profinite)} \\ \text{Hom}(\mathbb{Z}[T], \mathbb{Z}) & \text{if } i = 0 \end{cases}$
 $\Rightarrow \text{RHS} = \bigoplus_J \mathbb{Z}$. || $C(T, \mathbb{Z}) \cong \bigoplus_J \mathbb{Z}$.

• LHS: Use $0 \rightarrow \prod_J \mathbb{Z} \rightarrow \prod_J \mathbb{R} \rightarrow \prod_J \mathbb{R}/\mathbb{Z} \rightarrow 0$
 $\quad \quad \quad \downarrow \parallel_S$
 $\quad \quad \quad \mathbb{Z}[T]^{\text{ab}}$

• $\text{RHom}(\prod_J \mathbb{R}, \mathbb{Z}) \cong \text{RHom}_{\mathbb{R}}(\prod_J \mathbb{R}, \underbrace{\text{RHom}(\mathbb{R}, \mathbb{Z})}_{0 \text{ by Lisanne}}) = 0$

• $\text{RHom}(\prod_J \mathbb{R}/\mathbb{Z}) \cong \bigoplus_J \mathbb{Z}[-1]$
Remy

$\Rightarrow \text{LHS} = \bigoplus_J \mathbb{Z}$

(If follow the maps, then see that map is identity.)

Used by Dion.

□

Thm (i) • $\text{Solid} \subseteq \text{Cond}(\text{Ab})$ is abelian subcategory
stable under all limits, colimits, extensions

• $\{\prod_I \mathbb{Z}\}$ form family of comp. proj. generators
in Solid .

• Left adjoint $(-)^{\text{sol}} \dashv i$ if $A = \text{colim } \mathbb{Z}[S_i]$
then $A^{\text{sol}} = \text{cdim } \mathbb{Z}[S_i]^{\text{sol}}$

(ii) • $D(\text{Solid}) \leftrightarrow D(\text{Cond}(\text{Ab}))$ is fully faithful
with essential image all solid complexes.

• $C \text{ solid} \Leftrightarrow$ all $H^i(C)$ solid

• $(-)^{\text{Lsol}} \dashv i$ (Left derived solidification)

Strategy

① $\bigoplus \mathbb{Z}$ are solid (as complex)

② Complexes $\cdots \rightarrow \bigoplus \mathbb{Z} \rightarrow \cdots$ are solid

③ $\ker(\bigoplus \mathbb{Z} \rightarrow \bigoplus \mathbb{Z})$ are solid

④ $\text{Solid} \subseteq \text{Cond}(\text{Ab})$ is abelian subcategory

① Can prove

② Truncation arguments

③ and ④ follow from derived category arguments.

①

Goal Show $M = \bigoplus \pi \mathbb{Z}$ is solid

i.e. $\text{RHom}(\mathbb{Z}[S], \bigoplus \pi \mathbb{Z}) = \text{RHom}(\mathbb{Z}[S], \bigoplus \pi \mathbb{Z})$
(went to pull out \bigoplus)

- RHS: take S extremely disconnected, then $\mathbb{Z}[S]$ projective $\Rightarrow \text{RHom}(\mathbb{Z}[S], -) = \text{Hom}(\mathbb{Z}[S], -)$
- $\mathbb{Z}[S]$ compact $\Rightarrow \text{RHom}(\mathbb{Z}[S], -)$ commutes with filtered colimits.

$\Rightarrow \text{RHS} = \bigoplus \text{RHom}(\mathbb{Z}[S], \pi \mathbb{Z})$

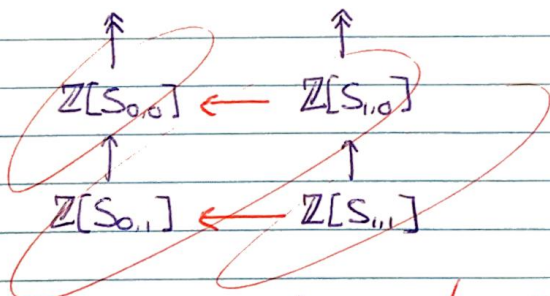
• LHS: Use $0 \rightarrow \prod_I \mathbb{Z} \rightarrow \prod_I \mathbb{R} \rightarrow \prod_I \mathbb{R}/\mathbb{Z} \rightarrow 0$.

Lemma $\text{RHom}(\prod_I \mathbb{R}/\mathbb{Z}, \bigoplus \pi \mathbb{Z}) = \bigoplus \text{RHom}(\prod_I \mathbb{R}/\mathbb{Z}, \pi \mathbb{Z})$

Proof Suffices to find resolution $\dots \rightarrow \mathbb{Z}[S_1] \rightarrow \mathbb{Z}[S_0] \rightarrow T \rightarrow 0$.
extremely disconnected.

Take Breen-Deligne:

$T \leftarrow \mathbb{Z}[T] \leftarrow \mathbb{Z}[T^2] \leftarrow \dots$



Take total complex.

Lemma $\text{RHom}(\mathbb{R}, \bigoplus \pi \mathbb{Z}) = 0 \quad (= \bigoplus \text{RHom}(\mathbb{R}, \pi \mathbb{Z}))$

Proof Use $0 \rightarrow \mathbb{Z} \rightarrow \mathbb{R} \rightarrow \mathbb{R}/\mathbb{Z} \rightarrow 0$.

- $\text{RHom}(\mathbb{Z}, M) = M$
- $\text{RHom}(\mathbb{R}/\mathbb{Z}, \bigoplus \pi \mathbb{Z}) \stackrel{\text{Just proven}}{=} \bigoplus \text{RHom}(\mathbb{R}/\mathbb{Z}, \pi \mathbb{Z})$

$\stackrel{\text{Remy}}{=} \bigoplus \pi \mathbb{Z}[-1] = M[-1] \quad \square$

Lemma $\text{RHom}(\prod_I \mathbb{R}, M) = 0$

" previous lemma

Proof $\text{RHom}_{\mathbb{R}}(\prod_I \mathbb{R}, \text{RHom}(\mathbb{R}, M)) = 0 \quad \square$

(3)

Proof Take $f: Y \rightarrow Z$ with kernel K .

Pick resolution $\cdots \rightarrow \overset{B}{\oplus \mathbb{Z}[s_j]} \rightarrow \oplus \mathbb{Z}[s_i] \rightarrow K \rightarrow 0$

Let $C = \left(\cdots \rightarrow \oplus \mathbb{Z}[s_j]^{\oplus} \rightarrow \oplus \mathbb{Z}[s_i]^{\oplus} \right)$

Now, $\text{RHom}(B, Y) = \text{RHom}(C, Y)$ and $\text{RHom}(B, Z) = \text{RHom}(C, Z)$

Hence,
$$\begin{array}{ccc} B & \xrightarrow{\sim} & K \\ \downarrow & \nearrow & \\ C & \xrightarrow{\exists!} & \end{array}$$
 so B is a retract of C .

Finally,

$$\text{RHom}(\mathbb{Z}[T]^{\oplus}, \overset{B}{\cancel{K}}) = \text{RHom}(\mathbb{Z}[T]^{\oplus}, \overset{B}{\cancel{K}})$$

↓
follows from \cong of C
and B is a retract of C .

□